Measure-theoretic Probability Note: Any Random Variable Plus Independent Gaussian Possesses Density

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Abstract

In this article, we demonstrate a well-established conclusion: when independent Gaussian noise ε is added to a random variable X, regardless of the distribution of X, discrete or continuous, deterministic or uncertain, the resulting random variable $X + \varepsilon$ will be absolutely continuous and possess a density function.

1 Preliminaries

Proposition 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{X}, \mathcal{X})$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{Y}, \mathcal{Y})$ be random variables. If X and Y are independent, then the joint law of X and Y, i.e. $P_{(X,Y)}$, equals the product measure $P_X \times P_Y$, where $P_X \coloneqq P \circ X^{-1}$ is the push-forward measure of X, and similarly for P_Y.*

Proof. For any rectangle $R \in \{A \times B : A \in \mathcal{X}, B \in \mathcal{Y}\}$,

$$\begin{aligned}
&P_{(X,Y)}(R) \\
&\quad = P(X \in A, Y \in B) \\
&\quad = P(X \in A)P(Y \in B) && \text{Independence of random variables} \\
&\quad = P_X(A)P_Y(B) \\
&\quad = (P_X \times P_Y)(R) && \text{Product measure of rectangles}
\end{aligned}$$

Thus, $P_{(X,Y)}$ and $P_X \times P_Y$ coincide on the set of rectangles. Then by the existence and uniqueness of product measure, they must agree on the entire product space $(\mathbb{X} \times \mathbb{Y}, \mathcal{X} \times \mathcal{Y})$. □

Definition 2 (Absolute continuity definition sketch)**.** *Let μ and π be two measures on Borel subsets. If for avery π-measurable set A, $\pi(A) = 0$ entails $\mu(A) = 0$, then we say μ is absolutely continuous w.r.t. π, or μ is dominated by π, denoted $\mu \ll \pi$.*

Definition 3 (Translation-invariance of measure)**.** *Let μ be a measure on $(\Omega, \mathcal{F})$. If for every $A \in \mathcal{F}$ and $x \in \Omega$ such that $A - x \coloneqq \{y \in \Omega : x + y \in A\} \in \mathcal{F}$, we have $\mu(A) = \mu(A - x)$, then we say μ is translation-invariant.*

Remark 3.1. *Lebesgue measure is the only non-zero measure up to scaling that is defined on Borel subsets, finite on compact sets, and translation-invariant.*

2 Main Result

Theorem 4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ be random variables. Let λ be a translation-invariant measure defined on $(\mathbb{R}, \mathcal{B})$. If X and Y are independent and $P_Y \ll \lambda$, then for $Z = X + Y$, $P_Z \ll \lambda$, and the density function of Z w.r.t. λ is written as $p_Z(z) = \int p_Y(z - x)\,\mathrm{d}P_X(x)$.*

Proof. Let A be any λ-measurable set. We have the following:

$$\begin{aligned}
P_Z(A) &= \int \mathbb{1}_A(Z(\omega))\,\mathrm{d}P(\omega) \\
&= \int \mathbb{1}_A(X(\omega) + Y(\omega))\,\mathrm{d}P(\omega) \\
&= \int \mathbb{1}_A(x + y)\,\mathrm{d}P_{(X,Y)}(x, y) && \text{Change of variable} \\
&= \int \mathbb{1}_A(x + y)\,\mathrm{d}(P_X \times P_Y)(x, y) && \text{Proposition 1} \\
&= \iint \mathbb{1}_A(x + y)\,\mathrm{d}P_Y(y)\,\mathrm{d}P_X(x) && \text{Fubini-Tonelli} \\
&= \iint \mathbb{1}_{A-x}(y)\,\mathrm{d}P_Y(y)\,\mathrm{d}P_X(x) \\
&= \int P_Y(A - x)\,\mathrm{d}P_X(x)
\end{aligned}$$

If $\lambda(A) = 0$, then by translation-invariance, $\lambda(A - x) = 0$; then by absolute continuity, $P_Y(A - x) = 0$. Hence $P_Z(A) = \int P_Y(A - x)\,\mathrm{d}P_X(x) = 0$. And therefore, $P_Z \ll \lambda$.

To find the density $p_Z(z)$,

$$\begin{aligned}
P_Z(A) &= \iint \mathbb{1}_A(x + y)\,\mathrm{d}P_Y(y)\,\mathrm{d}P_X(x) \\
&= \iint \mathbb{1}_A(x + y) p_Y(y)\,\mathrm{d}\lambda(y)\,\mathrm{d}P_X(x) && \text{Density of } P_Y \\
&= \iint \mathbb{1}_A(z) p_Y(z - x)\,\mathrm{d}(\lambda \circ f)(z)\,\mathrm{d}P_X(x) && \text{Change of variable } y = f(z) \\
&= \iint \mathbb{1}_A(z) p_Y(z - x)\,\mathrm{d}\lambda(z)\,\mathrm{d}P_X(x) && \text{Translation-invariance}
\end{aligned}$$

$$= \int_A \boxed{\int p_Y(z-x)\,\mathrm{d}P_X(x)}\,\mathrm{d}\lambda(z) \qquad \text{Tonelli}$$

□

Corollary 4.1. *Any random variable X with independent Gaussian noise added to it becomes absolutely continuous w.r.t. the Lebesgue measure and possesses a density function.*

Proof. By Remark 3.1 and the fact that Gaussian random variables are absolutely continuous, the conclusion follows. □

3 Case Study

In generative modeling that maximizes the model likelihood like variational autoencoder (P Kingma and Welling, 2014), an independent Gaussian noise is added to the model distribution:

$$X = f_{\text{decoder}}(Z) + \varepsilon\,, \quad \varepsilon \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})\,.$$

such that the law of X given Z, the latent, possesses a density function, which thus allows the model likelihood to be well-defined (Arjovsky and Bottou, 2017; Arjovsky et al., 2017). This is true no matter what distribution $f_{\text{decoder}}(Z)$ follows. Even if the push-forward measure of $f_{\text{decoder}}(Z)$ is already absolutely continuous (which rarely is), adding independent Gaussian noise to it simplifies the evaluation of its density. The downside, of course, is that the generated images X look blurry, the characteristics of Gaussian variables in images.

References

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